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On the variational characterization on conformally flat 3-manifolds(*)

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Abstract. *We review the characterization of conformally flat structures on a compact 3-manifold as the critical points of the Chern-Simons functional on Conformal Superspace. As a matter of fact, the formulas for the first and second variation of the Chern-Simons functional can essentially be found in the existing literature but we state the results more explicitly here and add some new remarks. In particular, pursuing our previous study* [17], *we want to keep track of the special properties of the De Witt metric.*

1. INTRODUCTION

Let M be a compact oriented manifold of dimension 3. Every 3-manifold admits a unique smooth structure [16]. Let $\mathscr{F}$ be the algebra of real functions on M, $\mathscr{W}$ the Weyl group of positive functions with pointwise multiplication, χ the Lie algebra of vector fields on M, Ω^p the space of p-forms on M, $\mathscr{S}$ the space of symmetric bilinear forms on M, $\mathscr{M}$ the open cone of Riemannian metrics of M in $\mathscr{S}$, and $\mathscr{D}$ the group of diffeomorphisms of M. We shall not make explicit the musical isomorphisms between χ and Ω^1 via g. By another abuse of notation, we shall not distinguish between vector fields and their pointwise values.

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Now $\mathscr{D}$ acts on $\mathscr{M}$ via the pull-back operation. In General Relativity, the resulting Moduli Space of Riemannian structures $\mathscr{M}/\mathscr{D}$ is called Superspace [7, 15]. On the other hand, $\mathscr{W}$ acts on $\mathscr{M}$ by pointwise multiplication. Let us define the group $\mathscr{C}$ of conformeomorphisms of M as the semi-direct product $\mathscr{C} = \mathscr{D} \ltimes \mathscr{W}$. The corresponding Moduli Space of conformal structures is called Conformal Superspace [8]. Both Superspace and Conformal Superspace are stratified orbifolds rather than genuine infinite-dimensional manifolds [3, 6, 7].

Superspace is the configuration space of the Hamiltonian description of General Relativity. In this context, it appears that the natural metric on Superspace is the DeWitt (= DW) metric $\langle . , . \rangle^{(DW)}$ which is defined as follows [17]. Identifying the tangent space of $\mathscr{M}$ at a generic metric g with $\mathscr{S}$, for $h, k \in \mathscr{S}$ one writes

$$(1.1) \qquad (h, k)_g^{(DW)} = g^{-2}(h \otimes k) - g^{-1}(h) g^{-1}(k)$$

for the local DeWitt scalar product. The DeWitt metric is the corresponding global scalarproduct which is obtained by integrating (1.1) with respect to the volume element of g. Actually the DeWitt "metric" is only a non-degenerate $\mathscr{D}$-invariant symmetric bilinear form on $\mathscr{M}$. Dropping the latter twisting term on the right hand side of (1.1), we recover the canonical positive-definite L^2 metric $\langle . , . \rangle$ on $\mathscr{M}$.

For a given $g \in \mathscr{M}$, let ∇ denote the Levi-Civita connection. We use the convention $\Delta = -\Sigma_i \nabla_i \nabla^i$ for the brute Laplacian. The divergence of a vector field X is given by $\delta X = -\Sigma_i \nabla_i X^i$. Let d stand for the exterior derivative on M. The Hessian Hess $= \nabla d$ maps $\mathscr{F} \to \mathscr{S}$.

We may couple ∇ and d as follows. Let E be any vector bundle on M. We consider the differential p-forms on M with values in E, i.e., the sections of $\Lambda^p M \otimes E$. We define the exterior differential d^∇ associated with ∇ by the following formula. For any section α of $\Lambda^p M \otimes E$, $d^\nabla \alpha$ is the section of $\Lambda^{p+1} M \otimes E$ such that for vector fields $X_0 \ldots, X_p$

$$(d^\nabla \alpha)(X_0, \ldots, X_p) =$$

$$= \sum_i (-1)^i \nabla_{X_i} (\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p))$$

$$+ \sum_{i \neq j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

The usual Riemann curvature tensor R of g is given by the formula

$$R_{X,Y} s = -d^\nabla(d^\nabla s)(X, Y)$$

for two vector fields $X, Y \in \chi$ and a section s of $E = TM$, the tangent bundle. Write ρ for the Ricci curvature and τ for the scalar curvature of g.

Let δ_g^* stand for the differential operator from χ to $\mathscr{S}$ which to a vector field X associates $\delta_g^* X = 1/2\, L_X g$. Write δ_g (resp. $\delta_g^{(DW)}$) for the formal adjoint of δ_g^* with respect to the canonical metric (resp. the DeWitt metric). Then δ_g is just the usual divergence, and $\delta_g^{(DW)} = \delta_g + dg^{-1}(.)$. The well-known decomposition theorem of Berger and Ebin [2] also holds for the DeWitt metric

$$\mathscr{S} = \operatorname{Im} \delta_g^* \oplus \operatorname{Ker} \delta_g^{(DW)}.$$

Geometrically, this means that the tangent space of $\mathscr{M}$ at g splits as an orthogonal sum with respect to the DeWitt metric. The image of δ_g^* contains the fundamental or vertical vector fields which are tangent to the orbit of $\mathscr{D}$ at g whereas the kernel of $\delta_g^{(DW)}$ consists of DeWitt horizontal vectors which are tangent to a transversal slice at g.

The gradient of any smooth $\mathscr{D}$-invariant functional with respect to the DeWitt metric on $\mathscr{M}$ is automatically DeWitt horizontal. For instance, minus the DeWitt gradient of the total scalar curvature functional yields a canonical DeWitt horizontal vector field on $\mathscr{M}$, to be called the DeWitt Einstein tensor

$$E^{(DW)} = \rho - \frac{1}{4}\tau g.$$

This is just the DeWitt twisting of the classical Einstein tensor

$$E = \rho - \frac{1}{2}\tau g.$$

Checking the horizontality directly amounts to applying the Bianchi identity.

Let us pause to introduce the Kulkarni-Nomizu product $\odot$ of two symmetric bilinear forms h and k

$$\begin{aligned}(h \odot k)(X, Y, Z, T) &= \\ &= h(X, Z)\, k(Y, T) + h(Y, T)\, k(X, Z) \\ &- h(X, T)\, k(Y, Z) - h(Y, Z)\, k(X, T).\end{aligned}$$

Remarkably, all curvature in 3 dimensions is given by

$$R = E^{(DW)} \odot g. \tag{1.2}$$

On the other hand, let us denote by $(h \circ k)_{ij} = \Sigma_l (h_{il} k^l_j + h_{jl} k^l_i)$ the symmetric product $\circ : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$.

Taking into account the action of the Weyl group, the Berger-Ebin decomposition can be refined so as to yield the York decomposition [8, 15, 21, 22]

$$\mathcal{S} = \operatorname{Im} P_g^* \oplus \mathcal{S}_g^C \oplus \mathcal{S}_g^{TT}.$$

Here $P_g^*X = \delta_g^*X + 1/3\ (\delta X) \cdot g$ for $X \in \chi$ is the linearization of the action of the conformeomorphism group or the trace-free part of δ_g^*.

The image of P_g^* may again be called vertical. In General Relativity literature, the component $\mathcal{S}_g^C$ consisting of the pure trace directions $\mathcal{F} \cdot g$ is called longitudinal whereas the vectors in the remaining part $\mathcal{S}^{TT}$ are referred to as the transverse traceless directions. A deformation $h \in \mathcal{S}$ of a metric $g \in \mathcal{M}$ is transverse traceless if and only if it is both divergence-free ($\delta_g h = 0$) and trace-free ($g^{-1}(h) = 0$). These are the physically interesting deformations which change the conformal structure in Conformal Superspace.

On a 3-manifold, there exist a canonical transverse traceless tensor called the York curvature which we want to describe next.

Obviously, any symmetric bilinear form $h \in \mathcal{S}$ can be thought as an element of $\Omega^1 \otimes \Omega^1$. We shall denote this identification by a tilde, $\tilde{h} \in \Omega^1 \otimes \Omega^1$. In particular, it makes sense to write $d^\nabla\ \tilde{h} \in \Omega^2 \otimes \Omega^1$. The bilinear forms in the kernel of this operation are called Codazzi tensors.

Let $*$ be the Hodge star of g. The codifferential $\delta^\nabla = -* d^\nabla *$ coincides on $\mathcal{S}$ with the divergence δ_g. The York curvature Y is given by

$$Y = * d^\nabla\ \tilde{E}^{(DW)}.$$

This is indeed a symmetric and transverse traceless 2-tensor.

THEOREM 1.3. (LOVELOCK [13]). *On a 3-manifold, the York curvature is the unique transverse traceless symmetric bilinear form which is a concomitant of the Riemannian metric and its derivatives up to order 3.* ■

The York curvature arose in a study of the initial-value problem in General Relativity [14, 15, 21, 22]. On the other hand, the 3-tensor containing third order derivatives in the Riemannian metric

$$C = d^\nabla\ \tilde{E}^{(DW)}$$

is well-known in mathematics literature. This is the classical Cotton tensor [5] whose non-vanishing is the obstruction for a Riemannian 3-manifold of being conformally flat. In higher dimensions, the analogous obstruction is the non-vanishing of the Weyl tensor, a component in the orthogonal decomposition of the Riemann curvature tensor.

Recall that conformal flatness means that every point of (M, g) admits a neighbourhood where g can be conformally rescaled to be flat. First examples of such manifolds are given by constant curvature manifolds and by Riemannian products of two constant curvature manifolds with opposite curvature signs. Moreover, a connected sum of two conformally flat manifolds again admits a conformally flat structure. As for simply-connected examples, there only exists the canonical sphere up to conformal equivalence (Kuiper [11]).

Furthermore, the Cotton tensor itself is a conformal invariant; in other words, it is fully invariant under a conformal rescaling $g \to \lambda \cdot g$ with $\lambda \in \mathscr{W}$. On the other hand, the Hodge star scales with a certain conformal weight so that the York curvature will be merely a conformal covariant, more precisely

$$Y(\lambda . g) = \lambda^{-1/2}\ Y(g).$$

The following new observation gives a certain correspondence between the tangent spaces of Superspace and Conformal Superspace at a constant curvature structure.

PROPOSITION 1.4. *If $h \in \mathscr{S}$ is DeWitt horizontal with respect to a metric g, then $* d^\nabla \tilde{h}$ is symmetric and traceless. Moreover, if $h \in \mathscr{S}$ is DeWitt horizontal with respect to a constant curvature metric g, then $* d^\nabla \tilde{h}$ is transverse traceless.*

Proof To prove the first statement, we work in local coordinates separating the target index to the right by a comma. Supposing that h is DeWitt horizontal, we find for instance

$$(* d^\nabla \tilde{h})_{1,2} = \nabla_2 h_{32} - \nabla_3 h_{22} = \nabla_3 h_{11} - \nabla_1 h_{31} = (* d^\nabla \tilde{h})_{2,1}$$

so that $* d^\nabla \tilde{h}$ is symmetric. Moreover, the image of $* d^\nabla$ is always traceless.

To prove the second assertion, we observe that

$$\delta_g * d^\nabla \tilde{h} = - * d^\nabla * * d^\nabla \tilde{h} = * R . \tilde{h}.$$

In local coordinates, using (1.2), this works out as

$$(\delta_g * d^\nabla \tilde{h})^1 = \sum_i (h_{i2}\, \rho^i_3 - h_{i3}\, \rho^i_2).$$

The other components are found by cyclical permutation of indices. Asking them all to vanish is equivalent to asking the Ricci curvature to be proportional to the metric. This is the Einstein condition which in 3 dimensions is equivalent with constant curvature by (1.2). ∎

2. CONFORMALLY FLAT 3-MANIFOLDS AS THE CRITICAL POINTS OF THE CHERN-SIMONS FUNCTIONAL

The gradient of any smooth functional on Conformal Superspace is automatically transverse traceless. For instance, we might try to pick the L^1 norm of the Cotton tensor. This is indeed a conformally invariant scalar. However, it is not smooth because of the presence of a square root. It is natural to ask as in [13] whether the York curvature arises as the gradient of some smooth functional. The answer in the affirmative was given by Chern and Simons [4] whose results imply that the York curvature is the gradient of a functional on Conformal Superspace which arises in their theory of secondary characteristic classes.

Consider the $SO(3)$ oriented frame bundle $F(M) \xrightarrow{\pi} M$ on M equipped with the Levi-Civita connection 1-form θ. The entries of θ are just the classical Christoffel symbols. The first non-trivial Chern-Simons transgressive polynomial is the 3-form on $F(M)$

$$TP_1(\theta) = \frac{1}{4\pi^2} \operatorname{Trace}\left(\frac{2}{3}\theta^3 + \theta\, d\theta\right)$$

where the trace is taken in the Lie algebra of $SO(3)$.

Recall that according to a classical theorem of Stiefel [19], every 3-manifold is parallelizable, i.e., the bundle $F(M) \xrightarrow{\pi} M$ admits a global gauge. Fixing any global framing $\mathcal{C}$, the Chern-Simons polynomial will descend to M and we may consider the circle-valued functional

$$\Phi(\theta) = \int_M \frac{1}{2}\, TP_1(\theta) \qquad \text{mod } Z.$$

The functional Φ is not as such a topological invariant of 3-manifolds although it comes from the theory of characteristic classes. In fact, the right way to capture the topological information carried by the Chern-Simons polynomials is to build a suitably defined K-theory for them [18].

The following facts on the functional Φ are established in [4] with slightly different terminology:

THEOREM 2.1. *Φ is well-defined; that is, Φ is independent of the choice of the trivialization $\mathcal{C}$. Moreover, Φ is a conformal invariant, hence a functional on Conformal Superspace. The gradient of Φ exists and is given by*

$$\operatorname{grad}\Phi = -\frac{1}{8\pi^2}\, Y.$$

Thus, the critical points of Φ are precisely the conformally flat 3-manifolds. We want to put on record the second variation formula for Φ in a less entangled notation than what occurs in [10], page 154. From standard variational formulas we deduce:

PROPOSITION 2.2. *The Hessian of Φ on two deformations $h, k \in \mathcal{S}^{TT}$ at a critical point g is given by the formula*

$$\text{Hess}\,\Phi(h, k) = -\frac{1}{8\pi^2}\langle *((d^\nabla)'(h)\,E^{(DW)} + d^\nabla(E^{(DW)})'(h)), k\rangle$$

where, in normal coordinates,

$$((d^\nabla)'(h)\,E^{(DW)})_{ij,l} = \sum_\nu \left(E^{(DW)}_{i\nu}\,[jl, \nu](h) - E^{(DW)}_{j\nu}\,[il, \nu](h)\right)$$

with

$$[\alpha\beta, \gamma](h) = \frac{1}{2}(D_\alpha h_{\beta\gamma} + D_\beta h_{\alpha\gamma} - D_\gamma h_{\alpha\beta}),$$

and

$$(E^{(DW)})'(h) = \frac{1}{2}\Delta h + \frac{3}{2}(\rho \circ h) - \frac{3}{4}(\rho, h)_g\, g - \frac{3}{4}\tau h.$$ ∎

The flat torus is easily seen to be non-stable. Lafontaine [12] investigates the stability of compact hyperbolic structures of dimension at least 3 viewed as conformally flat structures. He identifies their deformations as the traceless Codazzi tensors. Moreover, the following result contrasts with the well-known Mostow Rigidity Theorem.

THEOREM 2.3 (LAFONTAINE [12]). *If a compact hyperbolic manifold admits a totally geodesic hypersurface then it is not rigid viewed as a conformally flat manifold.* ∎

The geometric idea of the proof of the above is due to Apanasov. His recent work [1] gives new evidence for his conjecture that the subspace of conformally flat structures in Conformal Superspace is *disconnected.* He constructs an exotic conformally flat structure which cannot be approximated by any known deformations of the hyperbolic structure.

3. PERSPECTIVES

This paper was motivated by the recent work of Floer [9] who constructed eight new homology invariants for homology 3-spheres by means of a Morse theoretic study of the Chern-Simons functional on the space of connections of a principal $SU(2)$ bundle over the 3-manifold under study. Above we have provided a preliminary sketch for a potential extension of his deep results to a purely Riemannian context, in other words, passing from gauge theory to gravitation. In another deep recent work, Witten [20] found the conformally flat structures as the solutions of his theory of 2 + 1 dimensional Chern-Simons gravity. His action principle is just Φ. The author thanks Professors B. Apanasov, J.P. Bourguignon, A. Floer, J. Lafontaine, and I.M. Singer for their suggestions for further research on Φ.

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